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BOUNDARY VALUE PROCESSES: ESTIMATION AND IDENTIFICATION

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Abstract. Recent results obtained for boundary value processes and the associated smoothing and identification problems are presented in this paper. Both lumped and distributed parameter models are considered. Some open problems are discussed and the fundamental mathematical difficulties that arise in studying nonlinear extensions of the proposed models are mentioned.

1. INTRODUCTION

In this paper we discuss some recent results and open problems on boundary value processes. Boundary value processes are stochastic processes satisfying (ordinary or partial) stochastic differential equations subject to (stochastic) boundary conditions. A special class of one-parameter boundary value processes, with emphasis on system theoretic aspects, has been introduced and analyzed by Krener ([17], [18]).

In section 2, we discuss some recent developments on smoothing and likelihood ratio formulas for such processes. Likelihood ratio can be used, of course, to obtain the maximum likelihood estimate of unknown system parameters for such models. The consistency property of the maximum likelihood estimate is discussed in section 3. The multiparameter case is treated in section 4. We define an elliptic boundary value process with the disturbance terms white in both space and time. We consider the existence of solution of such an equation and study the associated smoothing and likelihood ratio formulas. Section 5 is devoted to open problems and new research directions in the area of boundary value processes.

The suggestions relate to the problems in which the author is personally involved at present. The choice of topics is purely subjective and does not pretend to cover all reseach activities in this field at present.

2. TWO-POINT BOUNDARY VALUE PROCESSES

A two-point boundary value process (TPBVP) is the solution of the vector stochastic differential equation

$$dx_t = A(t)x_t\ dt + G(t)u(t) + B(t)dw_t \tag{2.1}$$

satisfying the boundary condition

$$V^0 x_0 + V^T x_T = v \tag{2.2}$$

where $\{w_t, t \geq 0\}$ is a p-dimensional Brownian motion with

$$Ew_t = 0, \quad Ew_t w_s^\star = \min(t, s)I \tag{2.3}$$

(with " $\star$ " denoting transpose). The input function $\{u(t), t \geq 0\}$ is assumed to be a (known) deterministic ℓ - dimensional vector, while v is an n-dimensional normally distributed random vector, independent of $\{w_t,\ 0 \leq t \leq T\}$, with

$$Ev = m_v, \quad E(v - m_v)(v - m_v)^\star = \Sigma_v. \tag{2.4}$$

$A(\cdot), B(\cdot)$ and $G(\cdot)$ are $n \times n$, $n \times p$ and $n \times \ell$ matrix–valued functions, respectively, which are assumed to be continuous on [0,T], while V^0 and V^T are constant $n \times n$ matrices. Note that $\{x_t\}$ is defined by means of mixed boundary conditions and is, therefore, not a Markov processes.

Such models occur naturally in many physical situations. See, for example, Adams, Willsky and Levy [3] and Riddle and Weinert [23]. We shall begin with this model and study smoothing and identification problems connected with them. Suppose that the measurement process $\{y_t,\, 0 \leq t \leq T\}$ is given by

$$\begin{aligned} dy_t &= C(t)x_t dt + dw_{0t} \\ y_0 &= 0 \end{aligned} \tag{2.5}$$

where y_t has dimension m and $\{w_{0t},\, t \geq 0\}$ is an m-dimensional Brownian motion, independent of w and v, with

$$Ew_{0t}w_{0s}^{\star} = \min(t,s)I \tag{2.6}$$

where $C(\cdot)$ is a continuous matrix-valued function. The extension to the situation where

$$Ew_{0t}w_{0s}^{\star} = \int_0^{t\wedge s} S(\sigma)d\sigma, \quad S(t) > 0$$

a.e. t, is obvious.

The smoothing problem for the model (2.1, 2.2, 2.5) consists in determining $\widehat{x}_t \triangleq E[x(t) \mid y_s,\, 0 \leq s \leq T]$. This problem was first solved by Adams, Willsky and Levy ([2], [3]) using the idea of the complementary process, introduced earlier by Weinert and Desai [27]. The derivation holds under the assumption that the covariance matrix Σ_v is invertible. In this thesis, Adams [1] extended the result to the case when Σ_v is singular. An alternative derivation, based on functional analysis and systems realization, has recently been given by Bagchi and Westdijk [9] which does not need the invertibility of Σ_v either. The smoother is given by the following set of equations :

$$\widehat{x}_t = x^u(t) + \widehat{x}_t^c \tag{2.7}$$

$$\dot{x}^u(t) = A(t)x^u(t) + G(t)u(t) \tag{2.8a}$$

$$V^0x^u(0) + V^Tx^u(T) = m_v \tag{2.8b}$$

$$d\widehat{x}_t^c = A(t)\widehat{x}_t^c dt + B(t)B(t)^{\star}\widehat{p}_t^c dt \tag{2.9a}$$

$$d\widehat{p}_t^c = C(t)^{\star}C(t)\widehat{x}_t^c dt - A(t)^{\star}\widehat{p}_t^c dt - C(t)^{\star}[dy_t - C(t)x^u(t)dt] \tag{2.9b}$$

$$V^0\widehat{x}_0^c + V^T\widehat{x}_T^c = \Sigma_v F^{-1\star}(\widehat{p}_0^c - \Phi(T)^{\star}\widehat{p}_T^c) \tag{2.9c}$$

$$\Phi(T)^{\star}V^{T\star}F^{-1\star}\widehat{p}_0^c + V^{0\star}F^{-1\star}\Phi(T)^{\star}\widehat{p}_T^c = 0 \tag{2.9}$$

where $\Phi(T) \triangleq \Phi(T,0)$, with $\Phi(t,s)$ the state transition matrix for $\dot{x}(t) = A(t)x(t)$, and $F \triangleq V^0 + V^T\Phi(T)$ is assumed to be invertible. As is to be expected, we need to solve a $2n$- dimensional boundary value problem to obtain the smoother for a system with n-dimensional state space.

The other question of importance for the model (2.1, 2.2, 2.5) is the evaluation of the likelihood ratio. Using a well-known result of Shepp [25], the following expression for the likelihood ratio may be readily derived [9]:

$$\begin{aligned} LF(y) = d^{-1/2}\exp\Bigg\{ &\frac{1}{2}\int_0^T [C(t)\widehat{x}_t^c, dy_t - C(t)x^u(t)dt] \\ &+ \int_0^T [C(t)x^u(t), dy_t - C(t)x^u(t)dt] \\ &+ \frac{1}{2}\int_0^T [C(t)x^u(t), C(t)x^u(t)]dt \Bigg\} \end{aligned} \tag{2.10}$$

where $d = \det(I+R)$ with "det" standing for the determinant of the operator $(I+R)$ and R is the linear operator from $L_2^m[0,T]$ into itself with

$$(Rf)(t) = \int_0^T r(t,u)f(u)du \tag{2.11a}$$

$$r(t,u) \triangleq C(t)E(x_t^c x_u^{c\star})C(u)^\star. \tag{2.11b}$$

The term in the exponential may be evaluated using the smoothing equations described above. But equation (2.10) is still not of much direct use, since $d^{-1/2}$ is not given in terms of the system parameters.

In Bagchi and Westdijk [9] this determinant term has been evaluated explicitly by using Krener factorization [12]. Using this factorization, we can write

$$(I+R)^{-1} = (I-L^\star)(I-L) \tag{2.12}$$

where L is a Volterra operator with kernel $l(t,s)$. It may be shown that $(L+L^\star)$ has finite trace, and from [10] we know that

$$\log(d^{-1/2}) = -\frac{1}{2}\ \mathrm{Tr}\ (L+L^\star) = -\frac{1}{2}\int_0^T \mathrm{Tr}\ L(t,t)dt. \tag{2.13}$$

To determine the kernel $L(t,s)$ explicitly, we consider the model (2.1) - (2.5) restricted to the subinterval $[0,t]$, for fixed t. We see readily that $\{x_s, 0 \le s \le t\}$ satisfies

$$x_s = x^u(s) + x_s^c \tag{2.14a}$$

$$dx_s^c = A(s)x_s^c ds + B(s)dw_s, \quad 0 \le s \le s \tag{2.14b}$$

$$V^0 x_0^c + V^t x_t^c = v_t \tag{2.14c}$$

$$dy_t = C(t)x_t dt + dw_{0t}, \qquad y_0 = 0, \quad 0 \le s \le t \tag{2.14d}$$

where the random vector v_t is normally distributed and is *independent* of $\{w_s, w_{0s}; 0 \le s \le t\}$. Therefore, we get the same model as before, but now restricted to $[0,t]$. Let

$$G(t) \triangleq R(t)(I(t)+R(t))^{-1} \tag{2.15a}$$

where $R(t) : L_2^n[0,t] \to L_2^n[0,t]$ is given by

$$(R(t)f)(s) = \int_0^t r(s,u)f(u)du,\ 0 \le s \le t \tag{2.15b}$$

and $I(t)$ is the identity map on $L_2^n[0,t]$. Let $g(t;s,u)$ be the kernel of the operator $G(t)$. It can be shown that [10, pp. 130–132]

$$l(t,s) \equiv g(t;t,s) \tag{2.16}$$

Applying the smoothing formulas obtained above to the restricted model (4.14), it has been shown in [9] that

$$\int_0^T \mathrm{Tr}\ l(t,t)dt = \int_0^T \mathrm{Tr}\ [C(t)\{F(t)^{-1}\}_{1,2}C(t)^\star]dt \tag{2.17}$$

where

$$
\begin{aligned}
&F(t) \triangleq W^0(t)e^{-\underline{A}t} + W^T(t)\\
&W^0(t) \triangleq \begin{pmatrix} \Pi_0(t) & -I \\ \Pi_1(t)^\star & 0 \end{pmatrix}, \quad W^T(t) \triangleq \begin{pmatrix} \Pi_1(t) & 0 \\ \Pi_2(t) & I \end{pmatrix}\\
&\underline{A} \triangleq \begin{pmatrix} A & BB^\star \\ C^\star C & -A^\star \end{pmatrix}\\
&\dot{\Pi}_0(t) = \Pi_1(t)B(t)B(t)^\star\Pi_1(t), \quad \Pi_0(T) = V^{0\star}\Sigma_v^{-1}V^0\\
&\dot{\Pi}_1(t) = \Pi_1(t)B(t)B(t)^\star\Pi_2(t) - \Pi_1(t)A(t), \quad \Pi_1(T) = V^{0\star}\Sigma_v^{-1}V^T\\
&\dot{\Pi}_2(t) = \Pi_2(t)B(t)B(t)^\star\Pi_2(t) - \Pi_2(t)A(t) - A(t)^\star\Pi_2(t), \quad \Pi_2(T) = V^{T\star}\Sigma_v^{-1}V^T.
\end{aligned}
$$

and $\{F(t)^{-1}\}_{1,2}$ is the (1,2)-th block matrix element of $F(t)^{-1}$. This gives us

$$
\begin{aligned}
LF(y) = \exp.\Big(&-\frac{1}{2}\int_0^T \mathrm{Tr}\ [C(t)\{F(t)^{-1}\}_{1,2}C(t)^\star]dt\\
&+\frac{1}{2}\int_0^T [C(t)\widehat{x}_t^c,\ dy_t - C(t)x^u(t)dt] + \int_0^T [C(t)x^u(t),\ dy_t - C(t)x^u(t)dt]\\
&+\frac{1}{2}\int_0^T [C(t)x^u(t), C(t)x^u(t)]dt\Big).
\end{aligned}
\tag{2.18}
$$

3. IDENTIFICATION OF TPBVP MODELS

Let us now specialize to the time-invariant situation. Suppose that the system matrix A contains unknown parameters, denoted by a parameter vector $\theta \in \Theta$, with Θ compact in $\mathbf{R}^k$. Suppose that we make n^2 independent experiments. Based on the resulting observations $y_2^1, y_t^2, \cdots, y_t^{n^2}; 0 \le t \le T$, we want to determine the maximum likelihood estimate of the unknown parameter θ. This problem has been studied by Aihara and Bagchi [5]. We assume, for simplicity, that $u(t) \equiv 0$. The log-likelihood functional is given by

$$
\begin{aligned}
L_n(y^1, y^2, \cdots, y^{n^2}; \theta) = &\frac{1}{2}\sum_{k=1}^{n^2}\Big(\int_0^T [C\widehat{x}_t^k(\theta), dy_t^k]\\
&-\int_0^T \mathrm{Tr}\ [C\{F(t;\theta)^{-1}\}_{1,2}C^\star]dt\Big)
\end{aligned}
\tag{3.1}
$$

where, for $k = 1, 2, \cdots, n^2$,

$$dx_t^k(\theta) = A(\theta)x_t^k(\theta) + Bdw_t^k \tag{3.2a}$$

$$V^0x_0^k + V^Tx_T^k = v^k \tag{3.2b}$$

$$dy_t^k = Cx_t^kdt + dw_{0t}^k \tag{3.2c}$$

$$\widehat{x}_t^k \triangleq E[x_t^k \mid y_s^k; 0 \le s \le T]. \tag{3.2d}$$

Using the approach proposed by Borkar and Bagchi [14], consistency of the maximum likelihood estimate for TPBVP models has been analyzed in [5]. Consistency is based on the two following lemmas, the proofs of which are omitted:

Lemma 3.1.

$$\frac{1}{n^2}\sum_{k=1}^{n^2} x_t^k \to 0 \quad \text{as } n \to \infty,\ \forall t \in [0,T] \text{ a.s.} \tag{3.3a}$$

and

$$\frac{1}{n^2}\sum_{k=1}^{n^2}(x_t^k)(x_t^k)^* \to E[x_t x_t^*] \quad \text{as } n \to \infty, \forall t \in [0,T] \text{ a.s} \tag{3.3b}$$

with $\{x_t, 0 \le t \le T\}$ the solution of (2.1) - (2.2) when $u(t) \equiv 0$. □

Lemma 3.2.

$$\frac{1}{n^2}\sum_{k=1}^{n^2}\int_0^T \left[\int_0^T h(t,s)dy_s^k, dy_t^k\right] = \frac{1}{n^2}\sum_{k=1}^{n^2}\int_0^T [C\widehat{x}_t^k, dy_t^k]$$

$$\to E\int_0^T [C\widehat{x}_t, Cx_t]dt + \text{ Tr } (L + L^*) \qquad \text{as } n \to \infty \text{ a.s} \tag{3.4}$$

where $h(t,s)$ is the kernel of $R(I+R)^{-1}$. □

The likelihood functional $L_n(y_\cdot^1, y_\cdot^2, \cdots, y_\cdot^{n^2}; \theta)$ is given by (3.1). Θ being compact, we can find for each ω in the sample space an element $\theta_n(\omega) \in \Theta$ such that

$$L_n(y_\cdot^1, y_\cdot^2, \cdots, y_\cdot^{n^2}; \theta_n) \ge L_n(y_\cdot^1, y_\cdot^2, \cdots, y_\cdot^{n^2}; \theta') \quad \text{for all } \theta' \in \Theta \tag{3.5}$$

that is,

$$\sum_{k=1}^{n^2} \log \frac{dp_y^\theta}{dp_w}(y_\cdot^k) \,\Big|_{\theta=\widehat{\theta}_n} \ge \sum_{k=1}^{n^2} \log \frac{dp_y^{\theta'}}{dp_w}(y_\cdot^k) \qquad \text{for all } \theta' \in \Theta.$$

Here dp_y^θ is the measure induced by $\{y_t(\theta), 0 \le t \le T\}$ on $\mathbf{C}^{\,m}[0,T]$, the $\mathbf{R}^{\,m}$-valued continuous functions on $[0,T]$, and p_w is the Wiener measure thereon. In particular, if the true value of the parameter is assumed to lie in Θ, we have for $\theta' = \theta_0$ with θ_0 the true value of the parameter :

$$\sum_{k=1}^{n^2} \log \frac{dp_y^\theta}{dp_y^{\theta_0}}(y_\cdot^k) \ge 0. \tag{3.6}$$

We call θ_n the maximum likelihood estimate (MLE) of θ_0 based on n^2 independent sample trajectories $y_\cdot^1, y_\cdot^2, \cdots, y_\cdot^{n^2}$.

We now make the following assumptions:

A-1 $\quad \sup_{\theta\in\Theta} \|A(\theta)\| \le C$

A-2 $\quad \sup_{\theta\in\Theta} \|\nabla_\theta A(\theta)\| \le C.$

Lemma 3.3. Under **(A-1)** and **(A-2)**

$$\frac{1}{n^2}\sum_{k=1}^{n^2}\left(\int_0^T [C\widehat{x}_t^k(\theta), dy_t^k] - \int_0^T \text{ Tr } [C\{F(t,\theta)^{-1}\}_{1,2}C^*]dt\right)$$

is uniformly continuous in $\theta \in \Theta$, uniformly in n, with probability 1. □

Combining these lemmas, the following results have been proved in [5]:

THEOREM 3.1. *Let M be the set of measure zero outside of which Lemma 3.3 holds. For each $\omega \notin \mathcal{M}$, letting $\widehat{\theta}_n$ to denote the maximum likelihood estimate of θ_0, the following limit holds :*

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n^2} \int_0^T \|C(\widehat{x}_t^k(\widehat{\theta}_n) - \widehat{x}_t^k(\theta_0)\|^2 dt$$

$$= \int_0^T \operatorname{Tr}\, [C\{F(t;\theta_0)^{-1}\}_{1,2}C^\star]dt \text{ a.s.} \qquad \square$$

Corollary. (Consistency.) For each sample path ω, let

$$D(\omega) = \left\{ \theta \in \Theta \mid \liminf_{n\to\infty} \frac{1}{n^2} \sum_{k=1}^{n^2} \int_0^T \|C(\widehat{x}_t^k(\theta) - \widehat{x}_t^k(\theta_0))\|^2 dt \right.$$

$$\left. = \int_0^T \operatorname{Tr}\, [C\{F(t;\theta_0)^{-1}\}_{1,2}C^\star]dt \right\}. \tag{3.7}$$

Then $\widehat{\theta}_n \to D(\omega)$ as $n \to \infty$. By this, we mean that $\inf_{\theta' \in D(\omega)} \mid \widehat{\theta}_n - \theta' \mid \to 0$ a.s. as $n \to \infty$. In particular, if $D(\omega) = \{\theta_0\}$, independent of ω, then

$$\widehat{\theta}_n \to \theta_0 \quad \text{as } n \to \infty \qquad \text{a.s.} \qquad \square$$

4. ELLIPTIC BOUNDARY VALUE PROCESSES

By elliptic boundary value processes we mean processes satisfying stochastic elliptic partial differential equations. The randomness comes through a white noise input term. One can think of this as the natural multiparameter extension of the TPBVP model discussed above. Bensoussan, who first studied smoothing problems for such models in [13], assumed that both the state and observation noises have nuclear covariances. This meant that the spatial component of the noises were not white. This assumption is essential for working in the "countably additive" probability theory setup. To study models where the disturbance terms are white in both time and space components are more delicate. Bagchi and Aihara [8] developed a mathematical theory for such models by combining the Lions-Magenes theory [20] for treating elliptic partial differential equations and the white noise theory of Gelfand and Vilenkin [15], and of Hida [16], for modelling the distributed and boundary noises. We outline the main ideas involved.

Hida defined white noise in [16] by extending the canonical (finitely additive) Gauss measure on $L_2(G), G \subset \mathbf{R}^n$, to a countably additive measure on the dual of a nuclear space $S \subset L_2(G)$. But the dual $S^\star$ is too big to have reasonably regular solutions of the resulting partial differential equations. This problem has been resolved in [8] by extending the measure to be countably additive on the dual of an appropriate Sobolev space. Let G be an open, bounded domain in $\mathbf{R}^3$ satisfying the cone property (see Adams [4, p. 66] for definition). Theorem 6.5.3. in [4] asserts that the injection $H^2(G) \to L_2(G)$ is Hilbert-Schmidt. From the Sazanov-Minlos theorem [24, Theorem 2, p. 225], for the canonical Gaussian measure with characteristic functional

$$C_s(0) = \exp\left(-\frac{1}{2} \mid \phi \mid^2\right), \qquad \phi \in W_s \triangleq H^2(G) \tag{4.1}$$

there corresponds a unique measure μ_s on $(W_s^\star, B_s)$ such that

$$C_s(\phi) = \int_{W_s^\star} \exp\left(i\langle \xi, 0\rangle_s d\mu_s(\xi)\right) \tag{4.2}$$

where $\langle\cdot,\cdot\rangle_s$ denotes the duality between W_s and $W_s^\star$, and $|\cdot|$ denotes the usual $L_2(G)$-norm. We define the "coordinate process" in $(W_s^\star, B_s, \mu_s)$ as a Gaussian white noise in G. Using W_b to denote $H^2(\partial G)$, we can similarly define a Gaussian white noise on ∂G .

Elliptic boundary value processes may now be mathematically fomulated as follows. Let A be the elliptic operator :

$$A\phi = -\sum_{i,j=1}^{3} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial \phi}{\partial x_j}\right) + a_0(x)\phi \tag{4.3}$$

where we assume that

$$\sum_{i,j=1}^{3} a_{ij}(x)\xi_i\xi_j \geq \alpha(\xi_1^2+\xi_2^2+\xi_3^2), \quad \alpha > 0 \text{ a.e. } x \in G \tag{4.4a}$$

$$a_0(x) \geq \alpha \quad \text{a.e. } x \in G, \quad a_0 \in L^\infty(G) \tag{4.4b}$$

$$a_{ij} \in C^1(\overline{G}) \tag{4.4c}$$

Consider the following stochastic nonhomogeneous elliptic partial differential equation

$$Au(x) = f(x) + n_s(x) \quad \text{in } G \tag{4.5a}$$

$$\frac{\partial u(x)}{\partial \nu_{A^\star}} = n_b(x) \quad \text{on } \partial G \tag{4.5b}$$

where n_s and n_b are mutually independent white noise processes in G and ∂G , respectively; $f \in L_2(G)$ and

$$\frac{\partial(\cdot)}{\partial \nu_{A^\star}} = \sum_{i,j=1}^{3} a_{ij}(x)\frac{\partial(\cdot)}{\partial x_i}\cos\,(n, x_i).$$

Denoting $\mathcal{D}(A) = \{\phi \mid \phi \in H^2(G) \text{ and } \partial\phi/\partial\nu_{A^\star} = 0 \text{ on } \partial G\}$ and $A \in \mathcal{L}(\mathcal{D}(A); L_2(G))$, (4.5a-b) may be rewritten as

$$(u, A^\star\phi) = (f,\phi) + \langle n_s,\phi\rangle_s - \langle n_b,\phi\rangle_b \quad \forall\phi \in \mathcal{D}(A) \cap H^3(G) \tag{4.6}$$

where $A^\star$ is the adjoint of A and ϕ appearing in $\langle n_b,\phi\rangle_b$ stands for the trace value of ϕ. We have to take the test function ϕ in $H^3(G)$ in order to ensure that the trace value of ϕ is in $H^2(\partial G)$, so that $\langle n_b,\phi\rangle_b$ may be meaningfully defined. Under some additional assumption, it has been shown in [8] that there exists a unique solution u of equation (4.5) such that

$$u \in L_2(W_s \times W_b; L_2(G)).$$

The elliptic boundary value process $u(x)$ being properly formulated, we can consider $u(x)$ as a signal process and study the corresponding smoothing problem. The mapping

$$u(\cdot;\omega) : \Omega \to L_2(G)$$

with $\Omega \triangleq W_s \times W_b$, induces a countably additive measure μ_u on $L_2(G)$. Consider now a distributed observation mechanism

$$y(x) = u(x) + n_0(x) \tag{4.7}$$

where n_0 now is a finitely additive Gaussian white noise in $L_2(G)$, as introduced by Balakrishnan [12], independent of n_s and n_b . Let u, n_0 and y be $L_2(G)$-valued mappings on $L_2(G) \times \Omega$ defined by

$$\begin{aligned} u(\cdot;h,\omega) &= u(\cdot;\omega) \\ n_0(\cdot;h,\omega) &= n_0(\cdot;h) = h(\cdot) \\ y(\cdot;h,\omega) &= u(\cdot;h,\omega) + n_0(\cdot;h,\omega); \quad (h,\omega) \in L_2(G) \times \Omega. \end{aligned}$$

Then the observation equation may be reexpressed in $L_2(G)$ as

$$y = u + n_0 \tag{4.8}$$

with $L_2(G) \times \Omega$ the common underlying sample space. Our objective is to obtain computable formula for the smoother $\widehat{u} \triangleq E[u \mid y]$. The basic result in signal estimation is the white noise version of the Kallianpur-Streibel formula.

Proposition 1. Given y, u and n_0 as in (4.8) ,

$$E[g(u) \mid y] = \frac{\displaystyle\int_{L_2(G)} g(u) \exp\left(-\frac{1}{2}\{\mid u \mid_{L_2(G)} -2(u,y)\} d\mu_u\right)}{\displaystyle\int_{L_2(G)} \exp\left(-\frac{1}{2}\{\mid u \mid^2_{L_2(G)} -2(u,y)\} d\mu_u\right)} \tag{4.9}$$

for any function g integrable on $(L_2(G), \mu_u)$. The smoothing formula has been obtained in [8] as follows. We consider suitable finite dimensional approximation of u and first derive the smoothing equations for this approximate system. In the limit, we can show that $\widehat{u}$ satisfies

$$\begin{aligned} &(\widehat{u}, A^\star\phi) = (\widehat{v}, \phi) + (\widehat{v}, \phi)_b + (f, \phi), \quad \forall\phi \in \mathcal{D}(A), \widehat{u} \in L_2(G) \\ &A^\star\widehat{v} = y - \widehat{u}, \qquad\qquad \widehat{v} \in \mathcal{D}(A^\star) = \mathcal{D}(A). \end{aligned} \tag{4.10}$$

Just as in the case of TPBVP model, we can here also obtain the likelihood ratio formula similar to equation (2.10). The basic formula from which it can be derived is given by:

Proposition 2. The measure μ_y induced by y on $L_2(G)$ is cylindrical and is absolutely continuous with respect to the Gauss measure μ_G thereon. The corresponding Radon-Nikodym derivative is given by

$$RN(h) = \int_{L_2(G)} \exp\left(-\frac{1}{2}\{\mid u \mid^2_{l_2(G)} -2(u,h)\} d\mu_u\right), \quad h \in L_2(G). \tag{4.11}$$

Taking the same finite dimensional approximation as in deriving the smoothing equation and passing to the limit, we obtain

$$LF(y) \triangleq RN(y) = \exp\left(\frac{1}{2}\{(y, \widehat{u}) + (f, \widehat{v}) - \text{ Tr } \log(I + Q)\}\right) \tag{4.12}$$

where the covariance operator $Q \in \mathcal{L}(L_2(G); L_2(G))$ is trace class and satisfies

$$(A^\star\phi_1, QA^\star\phi_2) = (\phi_1, \phi_2) + (\phi_1, \phi_2)_b, \quad \forall\phi_1, \phi_2 \in \mathcal{D}(A). \tag{4.13}$$

Again the expression (4.12) has little practical value, as Tr $\log(I + Q)$ cannot be obviously written in terms of the system parameters.

5. OPEN PROBLEMS AND FUTURE RESEARCH

Unlike the TPBVP model, the express Tr $\log(I + Q)$ in terms of the system parameters is still an open problem. As in the one-parameter case, one would be tempted to use the Krein factorization theorem for this purpose. Unfortunately, however, not many concrete results are known for Krein factorization in the multiparameter case. We briefly discuss some very recent results in this direction for covariance operators of 2-parameter random fields. Details may be found in Luesink and Bagchi [21].

Let $\mathcal{H} = L_2([0,T_1] \times [0,T_2])^n$ and consider a 2-parameter n-dimensional random field $\{x_{\underline{t}}; \underline{t} = (t_1,t_2) \in [0,T_1] \times [0,T_2]\}$. Let $R : \mathcal{H} \to \mathcal{H}$ be the covariance operator of the random field. That is,

$$(Rf)(\underline{t}) = \int_0^{T_1}\int_0^{T_2} r(\underline{t},\underline{s}) f(\underline{s}) d\underline{s} \tag{5.1}$$

with $r(\underline{t},\underline{s}) \triangleq$ cov $(x_{\underline{t}}, x_{\underline{s}})$ continuous and $\int_0^{T_1}\int_0^{T_2} \|r(\underline{t},\underline{s})\|^2 d\underline{s} d\underline{t} < \infty$. Clearly the operator R is Hilbert-Schmidt ($d\underline{s} \triangleq ds_1 ds_2$). Then we can prove the following:

THEOREM 5.1. *$(I+R)$ may be factored as*

$$(I+R) = (I-P_2)(I-P_1)(I-P_1^\star)(I-P_2^\star) \tag{5.2}$$

where P_1 and P_2 are given by

$$(P_1 f)(\underline{t}) = \int_0^{t_1}\int_0^{t_2} h(t_2;\underline{t};\underline{s}) f(\underline{s}) d\underline{s} \tag{5.3a}$$

$$(P_2 f)(\underline{t}) = \int_0^{t_1}\int_{t_2}^{T_2} h(s_2;\underline{t},\underline{s}) f(\underline{s}) d\underline{s} \tag{5.3b}$$

with $h(q;\underline{t},\underline{s})$ the unique solution of

$$h(q;\underline{t},\underline{s}) - \int_{s_1}^{t_1}\int_0^{q} l(\underline{t},\underline{z}) h(q;\underline{z},\underline{s}) d\underline{z} = -l(\underline{t},\underline{s}) \tag{5.4a}$$

$$L(\underline{t},\underline{s}) \triangleq g(t_1,\underline{t},\underline{s}) \tag{5.4b}$$

and $g(q;\underline{t},\underline{s})$ the unique solution of

$$g(q;\underline{t},\underline{s}) + \int_0^{q}\int_0^{T_2} r(\underline{t},\underline{z}) g(q;\underline{z},\underline{s}) d\underline{z} = r(\underline{t},\underline{s}). \tag{5.4c}$$

PROOF. See [21]. □

As in the one-parameter case, one can use this result to calculate log det $(I+R)$. In fact, it has been shown in [21] that

$$\log\det(I+R) = -\operatorname{Tr}(P_1+P_2+P_1^\star+P_2^\star) = -\int_0^{T_1}\int_0^{T_2} \operatorname{Tr} h(t_2;\underline{t},\underline{t}) d\underline{t}$$
$$= \int_0^{T_1}\int_0^{T_2} \operatorname{Tr} l(\underline{t},\underline{t}) d\underline{t}. \tag{5.5}$$

The factorization (5.2), however remarkable, has its limitations. The domain considered is a rectangle and integrations are done in one direction, keeping the other constant. This is, of course, inherent in the multiparameter problem. It is not yet clear as to how this formula may be used to express the likelihood ratio for elliptic boundary value process models in terms of the system parameters.

The Krein factorization in the one-parameter case is directly related to the filtering (and smoothing) problem. How is the 2-parameter Krein factorization related to the filtering and smoothing problems for 2-parameter random fields is being investigated at present. In the one-parameter case, the determinant term gives precisely the correction term when Itô integral has to be evaluated as the limit of band limited processes ([11], [6]). Another open question is to obtain the corresponding correction term for multiparameter stochastic integral. For a discussion on such correction term, see [7] and the references there. Finally, the likelihood ratio formula for isotropic random fields described by non-causal internal differential models ([26], [19]) is an interesting problem which require further investigation.

An entirely different question is to consider nonlinear boundary value processes. The results from deterministic nonlinear boundary value problems cannot be readily extended to the stochastic situation. The reason is that Itô integrals are defined only for non-anticipative integrands. If, instead of (2.1), we consider a nonlinear stochastic differential equation with the same boundary condition as (2.2), the resulting system is not even defined in the framework of standard Itô calculus. It may be interesting to investigate whether the recent theory of anticipative stochastic integrals [22] may be used to formulate such nonlinear stochastic boundary value problems mathematically in a correct manner. Another possibility may be to interpret this as the limit of solutions of approximate boundary value processes with "band-limited" disturbance terms. The question becomes even more involved when one tries to define nonlinear elliptic boundary value processes.

REFERENCES

1. M.B. Adams, *Linear estimation of boundary value stochastic processes*, Sc. D. Thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, Cambridge, MA, January 1983; also Report LIDS-TH-1295; Laboratory for Information and Decision Systems, M.I.T..
2. M.B. Adams, A.S. Willsky and B.C. Levy, *Linear estimation of boundary value stochastic processes - Part I: The role and construction of complementary models*, IEEE Transactions on Automatic Control, AC-29 (1984), 803–811.
3. M.B. Adams, A.S. Willsky and B.C. Levy, *Linear estimation of boundary value stochastic processes - Part II: 1-D smoothing problems*, IEEE Transactions on Automatic Control, AC–29 (1984), 811–822.
4. R.A. Adams, "Sobolev Spaces," Academic Press, New York, 1975.
5. S.I. Aihara and A. Bagchi, *Parameter identification for Gaussian boundary value processes*, in preparation.
6. A. Bagchi, *Approximation of Itô integrals arising in stochastic time-delayed systems*, SIAM J. on Control and Optimization, 22 (1984), 878–888.
7. A. Bagchi, *Cylindrical measures in Hilbert space and likelihood ratio for two-parameter signals in additive white noise*, Control - Theory and Advanced Technology, 1 (1985), 139–153.
8. A. Bagchi and S.I. Aihara, *Smoothing and identification for random fields*, in "Advances in Communication and Control systems, ed. N. DeClaris," New York Optimization Software Inc., 1988, pp. 3–16.
9. A. Bagchi and H. Westdijk, *Smoothing and likelihood ratio for Gaussian boundary value processes*, IEEE Transactions on Automatic Control (to appear).
10. A.V. Balakrishnan, "Stochastic Differential Systems," Lecture Notes in Economics and Mathematical Systems, no. 84, Berlin: Springer-Verlag, 1973.
11. A.V. Balakrishnan, *On the approximation of Itô integrals using band-limited processes*, SIAM J. on Control, **12** (1974), 237-251.
12. A.V. Balakrishnan, "Applied Functional Analysis," 2nd Edition, New York, Springer-Verlag, 1981.
13. A. Bensoussan, "L' identification & Le Filtrage," IRIA, Cahier, No. 1, 1969.
14. V. Borkar and A. Bagchi, *Parameter estimation in continuous-time stochastic processes*, Stochastics, 8 (1982), 193–212.
15. I.M. Gelfand and N. Ya. Vilenkin, "Generalized Functions," New York Academic Press, 1964 (English Transl.).
16. T. Hida, "Stationary Stochastic Processes," Math. Notes Princeton Princeton University Press, New Yersey, 1970.
17. A.J. Krener, *One-dimensional acausal linear systems*, in "Proceedings of the 18th Conference on Decision and Control," Fort Lauderdale, Florida, 1979.
18. A.J. Krener, *Acausal realization theory, Part I: linear deterministic systems*, SIAM J. on Control and Optimization, 25 (1987), 499–525.
19. B.C. Levy and J.M. Tsitsiklis, *A fast algorithm for linear estimation of two-dimensional isotropic random fields*, IEEE Transactions on Information Theory, IT-31 (1985), 535–644.
20. J.L. Lions and E. Magenes, "Nonhomogeneous Boundary Value Problems and Applications, Vols. 1&2," Springer-Verlag, Berlin, 1970.

21. R. Luesink and A. Bagchi, *Krein factorization of covariance operators in signal-plus-noise 2-parameter random field models and application to the likelihood ratio*, TW-Memorandum, Dept. of Applied Mathematics, Univ. of Twente, Enschede, The Netherlands (1989).
22. D.N. Nualart and M. Zakai, *Generalized stochastic integrals and the Malliavin calculus*, Prob. Theory and Related Fields 73 (1986), 255–280.
23. L.R. Riddle and H.L. Weinert, *Recursive linear smoothing for dissipative hyperbolic systems*, Mechanical Systems and Signal Processing, 2 (1988), 77–96.
24. L. Schwarz, "Radon Measures on Arbitrary Topological Spaces and Cylindrical Measures," Tata Institute of Fundamental Research, Oxford University Press, 1973.
25. L.A. Shepp, *Radon-Nikodym derivatives of Gaussian measures*, Annals of Mathematical Satistics, 37 (1966), 321–354.
26. A.H. Tewfik, B.C. Levy and A.S. Willsky, *Recursive estimation for 2-D isotropic random fields*, Report LIDS-P-1585, Laboratory for Information and Decision Systems, M.I.T. (1986).
27. H.L. Weinert and U.B. Desai, *On complementary models and fixed-interval smoothing*, IEEE Transactions on Automatic Control, AC-26 (1981), 863–867.